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Fisher zeros and singular behaviour of the two-dimensional Potts model in the thermodynamic limit

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Abstract. The duality transformation is applied to the Fisher zeros near the ferromagnetic critical point in the $q > 4$ state two-dimensional Potts model. A requirement that the locus of the duals of the zeros be identical to the dual of the locus of zeros in the thermodynamic limit (i) recovers the ratio of specific heat to internal energy discontinuity at criticality and the relationships between the discontinuities of higher cumulants and (ii) identifies duality with complex conjugation for the zeros near the ferromagnetic critical point. The conjecture that all zeros governing ferromagnetic singular behaviour satisfy the latter requirement gives the full locus of such Fisher zeros to be a circle. This locus, together with the density of zeros is then shown to be sufficient to recover the *singular* part of the thermodynamic functions in the thermodynamic limit, their regular parts coming from separate loci of zeros not crossing the positive real temperature axis.

1. Introduction

The q-state Potts model [1], introduced in 1952 as a generalization of the Ising model [2], has become the generic model for the analytical and numerical study of first- and second-order phase transitions [3]. Apart from the one-dimensional case [1], the only solution which exists to date is for the $q = 2$ (Ising) model in two dimensions and in the absence of an external magnetic field [4]. The partition function for the standard Potts model is $Z_L(\beta) = \sum_{\{\sigma_i\}} \exp(\beta \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j})$ where $\beta = 1/(k_B T)$ and T is the temperature. The spin σ_i at site i on a d-dimensional lattice takes values $1, 2, \ldots, q$ and the total number of sites is $V = L^d$. Despite the absence of a full solution for general q, some exact results are obtainable in $d = 2$ dimensions. The first of these is that, up to an irrelevant multiplicative constant, the form of the partition function is unchanged under a duality transformation in the thermodynamic limit [1]. In terms of the low-temperature expansion variable $u = \mathrm{e}^{-\beta}$, this duality transformation is $u \to \mathcal{D}(u)$, where

$$\mathcal{D}(u) = \frac{1-u}{1+(q-1)u}. \tag{1}$$

The critical temperature at which the phase transition occurs is invariant under (1) and is given by [1]

$$\mathcal{D}(u_c) = u_c \qquad \text{or} \qquad u_c = \frac{1}{1+\sqrt{q}}. \tag{2}$$

Baxter has shown that *at* the critical point the model is equivalent to a solvable homogeneous ice-type model [5, 6]. By deriving the latent heat at criticality it was shown that the phase transition in the two-dimensional model is first (second) order for $q > 4$ ($q \leqslant 4$). In fact, for the $q > 4$ case, the exact values of the latent heat, the mean internal energy and the specific heat discontinuity (but not, for example, the mean specific heat) are known [1, 5–7]. The full form of the free energy (and derivable thermodynamic functions) of the Potts model has, however, never been calculated for general q and general T. In the words of Baxter, solving the Potts model for general temperatures is, therefore, 'a very tantalising problem' [6]. The Potts model is reviewed in [7].

In this paper the problem is approached using a remarkably general and recently derived result concerning the partition function zeros of models with a first-order phase transition [8].

For finite systems the zeros of the partition function [9] are strictly complex (non-real). As $L \to \infty$ one expects these zeros to condense onto a smooth curve whose impact on to the real parameter axis precipitates the phase transition. Knowledge of the locus and density of partition function zeros is sufficient to determine the full thermodynamic behaviour of the system. Fisher [10] emphasized the application of zeros in the complex temperature plane to the study of temperature-driven phase transitions. In particular, in [10], the Kaufman solution [11] of the two-dimensional Ising model was used to show that the Fisher zeros (also called complex temperature zeros [12]) are dense on two circles in the complex u-plane in the thermodynamic limit.

The question of the locus of Fisher zeros in the $d = 2$ Potts model, in particular, is one which has recently received an increased amount of attention (see [8, 12–17]). Based on similarities with the Ising case, Martin [13] and Maillard and Rammal [14] conjectured that the locus of Fisher zeros in the $d = 2$, q-state Potts model can be given by an extension of (2) to the complex plane, namely $\mathcal{D}(u) = u^*$, where u^* is the complex conjugate of u, although on this basis alone it was 'not clear where this requirement comes from' [13]. In section 2 the origin of this identification is explained. This identification yields a circle with centre $-1/(q-1)$ and radius $\sqrt{q}/(q-1)$. When $q = 2$ this recovers the so-called ferromagnetic Fisher circle of the Ising model [10]. In the Ising case, the partition function is actually a function of u^2. There, the second (so-called antiferromagnetic) Fisher circle comes from the map $u \to u^{-1}$ ($\beta \to -\beta$). Numerical investigations for small lattices at $q = 3$ and 4 [13, 14] provided evidence that the Fisher zeros do indeed lie on the circle given by the identification of duality with complex conjugation. However, the numerics are highly sensitive to the boundary conditions used and the situation far from criticality remained unclear. Some progress was made recently in the non-critical region using low-temperature expansions for $3 \leqslant q \leqslant 8$ [12].

Recently, and on the basis of numerical results on small lattices (up to $L = 7$) with $q \leqslant 10$, it has again been conjectured that for finite lattices with self-dual boundary conditions, and for other boundary conditions in the thermodynamic limit, the zeros in the ferromagnetic regime are on the above circle [15]. The conjecture of [15] was, in fact, proven for infinite q in [16]. This circle conjecture is similar to another recent conjecture [17], namely that the Fisher zeros for the q-state Potts model on a triangular lattice with pure three-site interaction in the thermodynamic limit (which is also self-dual [18]) lie on a circle and a segment of the negative real axis.

All of the above conjectures regarding the locus of Fisher zeros are, at least in part, numerically based. In this paper, the problem is addressed analytically. A requirement that taking the thermodynamic limit and application of the duality transformation to the Fisher zeros be commutative in the $q > 4$ case (i) recovers the ratio of specific heat discontinuity to latent heat and corresponding relationships between the discontinuities of higher cumulants and (ii) analytically identifies duality with complex conjugation for the zeros near ferromagnetic criticality. Conjecturing that all zeros governing ferromagnetic singular behaviour satisfy the latter requirement, the locus of such Fisher zeros is shown to indeed be a circle. This locus, together with the density of zeros is then shown to be *sufficient* to recover the singular form of all thermodynamic functions in the thermodynamic limit. It is therefore expected that the regular behaviour of these thermodynamic functions is governed by separate (undetermined) loci of zeros not crossing the positive real temperature axis.

2. Thermodynamic functions

For finite L, the partition function can be written as a polynomial of finite degree in u, and as such, can be expressed in terms of its complex Fisher zeros $u_j(L)$ [9] as $Z_L(\beta) \propto \prod_{j=1}^{dV}(u - u_j(L))$. The free energy is defined by $\beta f(\beta) = -\ln Z(\beta)/V$. The internal energy is therefore

$$e(\beta) = \frac{\partial(\beta f)}{\partial \beta} = \text{constant} + \frac{u}{V}\sum_{j=1}^{dV}\frac{1}{u - u_j(L)}. \tag{3}$$

The specific heat and the general nth cumulant are, respectively, defined as

$$c(\beta) = -k_B\beta^2\frac{\partial^2(\beta f)}{\partial\beta^2} \qquad \gamma_n(\beta) = (-)^{n+1}\frac{\partial^n(\beta f)}{\partial\beta^n}. \tag{4}$$

Using the notation

$$\Delta\gamma_n \equiv \lim_{\beta\nearrow\beta_c}\gamma_n(\beta) - \lim_{\beta\searrow\beta_c}\gamma_n(\beta) \tag{5}$$

for the discontinuity in the nth cumulant at the critical temperature, the exact results [1, 5–7] (in the thermodynamic limit) are

$$\overline{e} \equiv \tfrac{1}{2}\left(\lim_{\beta\nearrow\beta_c} e(\beta) + \lim_{\beta\searrow\beta_c} e(\beta)\right) = -\left(1 + \frac{1}{\sqrt{q}}\right) \tag{6}$$

$$\Delta e = 2\left(1 + \frac{1}{\sqrt{q}}\right)\tanh\left(\frac{\Theta}{2}\right)\prod_{n=1}^{\infty}\tanh^2(n\Theta) \tag{7}$$

$$\Delta c = k_B\beta_c^2\frac{\Delta e}{\sqrt{q}} \tag{8}$$

where $\Theta = \ln\left(\sqrt{q/4} + \sqrt{q/4 - 1}\right)$. Further results include the general higher cumulant combination $\gamma_n(\beta_c^-) - (-)^n\gamma_n(\beta_c^+)$ determinable from duality [7, 19, 20].

3. Partition function zeros

Recently, Lee [8] has derived a general theorem for first-order phase transitions in which the partition function zeros can be expressed in terms of the discontinuities in the thermodynamic functions (for finite size as well as in the infinite volume limit). One may recover Lee's result for the q-state Potts model with large q from the following rigorous result: in a finite volume $V = L^d$ with periodic boundary conditions and close to the transition point β_c, the partition function for the Potts model with a disordered phase of free energy $f_d(\beta)$ and a q-fold degenerate ordered phase of free energy $f_o(\beta)$ is [21]

$$Z_L(\beta) = \mathrm{e}^{-V\beta f_d(\beta)} + q\mathrm{e}^{-V\beta f_o(\beta)} + \mathcal{O}\left(\mathrm{e}^{-bL}\right)\mathrm{e}^{-V\beta\min\{f_d(\beta), f_o(\beta)\}} \tag{9}$$

in which b is a positive constant. This partition function is zero when

$$\beta(f_d(\beta) - f_o(\beta)) = -\frac{\ln q + \mathcal{O}\big(\mathrm{e}^{-bL}\big)}{V} \pm \mathrm{i}\frac{2j-1}{V}\pi \tag{10}$$

for integer j. Expanding the left-hand side around β_c and dropping the $\mathcal{O}\big(\mathrm{e}^{-bL}\big)$ term, one finds†

$$\frac{\ln q}{V\Delta e} \pm \mathrm{i}\frac{(2j-1)\pi}{V\Delta e} = \beta_c t + \frac{t^2}{2!}\frac{\Delta c}{k_B\Delta e} + \sum_{n=3}^{\infty}\frac{(\beta_c t)^n}{n!}\frac{\Delta\gamma_n}{\Delta e} \tag{11}$$

where the reduced temperature is $t = 1 - \beta/\beta_c$. This is the same result as that of Lee [8] for a system with a temperature-driven phase transition. Inverting, we find [8]

$$\begin{aligned} \beta_c \operatorname{Re} t_j(L) &= A_1\hat{I}_j^2 + A_3\hat{I}_j^4 + A_5\hat{I}_j^6 + \cdots + \mathcal{O}(1/V) \\ \pm\beta_c \operatorname{Im} t_j(L) &= \hat{I}_j + A_2\hat{I}_j^3 + A_4\hat{I}_j^5 + \cdots + \mathcal{O}(1/V) \end{aligned} \tag{12}$$

where $\hat{I}_j = (2j-1)\pi/(V\Delta e)$ and $\mathcal{O}(1/V)$ represents terms which vanish in the infinite volume limit and where the coefficients A_n are easily calculable, the first few being [8]

$$A_1 = \frac{\Delta c}{2k_B\beta_c^2\Delta e} \tag{13}$$

$$A_2 = -2A_1^2 + \frac{\Delta\gamma_3}{3!\Delta e} \tag{14}$$

$$A_3 = -5A_1^3 + 5A_1\frac{\Delta\gamma_3}{3!\Delta e} - \frac{\Delta\gamma_4}{4!\Delta e} \tag{15}$$

$$A_4 = 14A_1^4 - 21A_1^2\frac{\Delta\gamma_3}{3!\Delta e} + 3\left(\frac{\Delta\gamma_3}{3!\Delta e}\right)^2 + 6A_1\frac{\Delta\gamma_4}{4!\Delta e} - \frac{\Delta\gamma_5}{5!\Delta e} \tag{16}$$

$$\begin{aligned} A_5 = 42A_1^5 &- 84A_1^3\frac{\Delta\gamma_3}{3!\Delta e} + 28A_1\left(\frac{\Delta\gamma_3}{3!\Delta e}\right)^2 + 28A_1^2\frac{\Delta\gamma_4}{4!\Delta e} - 7\frac{\Delta\gamma_3}{3!\Delta e}\frac{\Delta\gamma_4}{4!\Delta e} \\ &-7A_1\frac{\Delta\gamma_5}{5!\Delta e} + \frac{\Delta\gamma_6}{6!\Delta e}. \end{aligned} \tag{17}$$

From (12) the real part of the zeros (in the thermodynamic limit) can be expressed in terms of their imaginary parts as $\beta_c \operatorname{Re} t = \mathcal{L}(\beta_c \operatorname{Im} t)$ where

$$\mathcal{L}(\theta) = A_1\theta^2 + (-2A_1A_2 + A_3)\theta^4 + (7A_1A_2^2 - 2A_1A_4 - 4A_2A_3 + A_5)\theta^6 + \cdots. \tag{18}$$

The zeros are thus seen to lie on a curve. In the complex u upper half-plane the equation of this curve is

$$\gamma^{(+)}(\theta) = u_c \mathrm{e}^{\mathcal{L}(\theta) + \mathrm{i}\theta}. \tag{19}$$

This defines the locus of zeros in the infinite volume limit.

† The author thanks Wolfhard Janke for pointing out this argument.

3.1. The dual of the locus of zeros and the locus of the duals of the zeros

Applying the duality transformation (1) to $\gamma^{(+)}(\theta)$ and expanding in powers of θ gives the dual of the locus $\mathcal{D}\left(\gamma^{(+)}(\theta)\right)$:

$$\begin{aligned}\mathrm{Re}\,\mathcal{D}\left(\gamma^{(+)}(\theta)\right) = u_c\bigg[1 + \frac{\theta^2}{2q}(2\sqrt{q} - 2A_1 q - q) - \frac{\theta^4}{24q^2}\big(24\sqrt{q} - 36q + 14q^{3/2} - q^2 \\ -72A_1 q + 72A_1 q^{3/2} - 12A_1 q^2 + 24A_1^2 q^{3/2} - 12A_1^2 q^2 \\ -48A_1 A_2 q^2 + 24A_3 q^2\big) + \cdots\bigg]\end{aligned} \tag{20}$$

$$\begin{aligned}\mathrm{Im}\,\mathcal{D}\big(\gamma^{(+)}(\theta)\big) = -u_c\bigg[\theta - \frac{\theta^3}{6q}\big(6 - 6q^{1/2} + q - 12A_1 q^{1/2} + 6A_1 q\big) \\ -\frac{\theta^5}{120q^2}\big(-120 + 240q^{1/2} - 150q + 30q^{3/2} - q^2 + 480A_1 q^{1/2} - 720A_1 q \\ +280A_1 q^{3/2} - 20A_1 q^2 - 360A_1^2 q + 360A_1^2 q^{3/2} - 60A_1^2 q^2 + 480A_1 A_2 q^{3/2} \\ -240A_1 A_2 q^2 - 240A_3 q^{3/2} + 120A_3 q^2\big) + \cdots\bigg].\end{aligned} \tag{21}$$

Alternatively, applying the duality transformation (1) directly to the jth zero in the finite-size system and expanding again, gives

$$\begin{aligned}\beta_c\,\mathrm{Re}\,t_j^D(L) = A_1^D \hat{I}_j^2 + A_3^D \hat{I}_j^4 + A_5^D \hat{I}_j^6 + \cdots + \mathcal{O}(1/V) \\ \mp\beta_c\,\mathrm{Im}\,t_j^D(L) = \hat{I}_j + A_2^D \hat{I}_j^3 + A_4^D \hat{I}_j^5 + \cdots + \mathcal{O}(1/V)\end{aligned} \tag{22}$$

where the first few coefficients A_n^D are

$$A_1^D = q^{-1/2} - A_1 \tag{23}$$
$$A_2^D = -q^{-1} + 2q^{-1/2}A_1 + A_2 \tag{24}$$
$$A_3^D = -\tfrac{1}{12}q^{-1/2} - q^{-3/2} + 3q^{-1}A_1 - q^{-1/2}A_1^2 + 2q^{-1/2}A_2 - A_3 \tag{25}$$
$$\begin{aligned}A_4^D = \tfrac{1}{4}q^{-1} + q^{-2} - \tfrac{1}{3}q^{-1/2}A_1 - 4q^{-3/2}A_1 + 3q^{-1}A_1^2 - 3q^{-1}A_2 + 2q^{-1/2}A_1A_2 \\ +2q^{-1/2}A_3 + A_4\end{aligned} \tag{26}$$
$$\begin{aligned}A_5^D = \tfrac{1}{360}q^{-1/2} + \tfrac{1}{2}q^{-3/2} + q^{-5/2} - \tfrac{5}{4}q^{-1}A_1 - 5q^{-2}A_1 + 6q^{-3/2}A_1^2 + \tfrac{1}{2}q^{-1/2}A_1^2 - q^{-1}A_1^3 \\ -\tfrac{1}{3}q^{-1/2}A_2 - 4q^{-3/2}A_2 + 6q^{-1}A_1A_2 + q^{-1/2}A_2^2 + 3q^{-1}A_3 - 2q^{-1/2}A_1A_3 \\ +2q^{-1/2}A_4 - A_5.\end{aligned} \tag{27}$$

From (22) the real part of the dual zeros can be expressed in terms of their imaginary parts in the thermodynamic limit as $\beta_c\,\mathrm{Re}\,t_j^D = \mathcal{L}^D(\mathrm{Im}\,t_j^D)$ where

$$\mathcal{L}^D(\theta) = A_1^D\theta^2 + (-2A_1^D A_2^D + A_3^D)\theta^4 + (7A_1^D A_2^{D^2} - 2A_1^D A_4^D - 4A_2^D A_3^D - A_5^D)\theta^6 + \cdots. \tag{28}$$

Therefore, the locus of the duals of the upper half-plane zeros in the thermodynamic limit is given by

$$\gamma^{(+)^D}(\theta) = u_c \mathrm{e}^{\mathcal{L}^D(\theta) - \mathrm{i}\theta}. \tag{29}$$

The expansion of this locus of duals is

$$\mathrm{Re}\,\gamma^{(+)D}(\theta) = u_c\left[1 + \frac{\theta^2}{2!}\left(-1 + 2A_1^D\right)\right.$$
$$\left. + \frac{\theta^4}{4!}\left(1 - 12A_1^D - 48A_1^D A_2^D + 24A_3^D + 12A_1^{D2}\right) + \cdots\right] \tag{30}$$

$$\mathrm{Im}\,\gamma^{(+)D}(\theta) = -u_c\left[\theta + \frac{\theta^3}{3!}\left(-1 + 6A_1^D\right)\right.$$
$$\left. + \frac{\theta^5}{5!}\left(1 - 20A_1^D + 60A_1^{D2} - 240A_1^D A_2^D + 120A_3^D\right) + \cdots\right]. \tag{31}$$

In deriving the dual of the locus of zeros (20) and (21), the duality transformation was applied *after* the thermodynamic limit of the positions of the zeros (i.e. their locus) was taken. In equations (30) and (31) the duality transformation was applied to the zeros *before* taking the thermodynamic limit. Even in the case where the finite-L system does not have duality-preserving boundary conditions, taking the thermodynamic limit restores self-duality. The dual of the (thermodynamic limit) locus of zeros and the (thermodynamic limit) locus of the duals of the zeros must be identical. We therefore demand, that

$$\mathcal{D}\left(\gamma^{(+)}\right) \equiv \gamma^{(+)D} \tag{32}$$

order by order in the expansion in θ. Up to $\mathcal{O}(\theta^2)$ this is trivial. To $\mathcal{O}(\theta^3)$ and (separately at) $\mathcal{O}(\theta^4)$ they are identical if $A_1 = 1/(2\sqrt{q})$. From (13), this is the result (8). The identity (32) at $\mathcal{O}(\theta^5)$ and (separately at) $\mathcal{O}(\theta^6)$ gives $A_3 = A_2/\sqrt{q} - q^{-3/2}(q-3)/24$, which from (14) and (15) means that

$$\Delta\gamma_4 = \frac{6}{\sqrt{q}}\Delta\gamma_3 + \frac{q-6}{q^{3/2}}\Delta e. \tag{33}$$

Higher-order results are obtainable using a computer algebra system such as Maple. To orders $\mathcal{O}(\theta^7)$ and $\mathcal{O}(\theta^8)$ and (separately) to orders $\mathcal{O}(\theta^9)$ and $\mathcal{O}(\theta^{10})$ one finds

$$\frac{\Delta\gamma_6}{6!\Delta e} = \frac{5}{2q^{1/2}}\frac{\Delta\gamma_5}{5!\Delta e} + \frac{q-20}{8q^{3/2}}\frac{\Delta\gamma_3}{3!\Delta e} + \frac{1}{6!q^{1/2}} - \frac{1}{8q^{3/2}} + \frac{1}{2q^{5/2}} \tag{34}$$

and

$$\frac{\Delta\gamma_8}{8!\Delta e} = \frac{7}{2q^{1/2}}\frac{\Delta\gamma_7}{7!\Delta e} + \left(\frac{5}{24q^{1/2}} - \frac{35}{4q^{3/2}}\right)\frac{\Delta\gamma_5}{5!\Delta e} + \left(\frac{3}{6!q^{1/2}} - \frac{15}{16q^{3/2}} + \frac{21}{2q^{5/2}}\right)\frac{\Delta\gamma_3}{3!\Delta e}$$
$$+ \frac{1}{8!q^{1/2}} - \frac{23}{960q^{3/2}} + \frac{5}{8q^{5/2}} - \frac{17}{8q^{7/2}} \tag{35}$$

respectively. These results and further results for the higher cumulants at criticality are also obtainable directly from the duality transformation (33) (see [7, 19, 20]).

3.2. The full ferromagnetic locus of zeros

Putting the above equations into (23)–(27) (and their higher-order equivalents) yields $A_j^D = A_j$ (this has been verified up to $j = 8$). Therefore (at least up to θ^{10}) the dual of the locus of zeros is the complex conjugate of the original locus of zeros. We now assume that this is the case for all θ. Then, the full ferromagnetic locus of zeros (that part of the full locus which intersects the real temperature axis at the physical ferromagnetic critical point) is found by identifying [13, 14]

$$\mathcal{D}\left(\gamma^{(+)}(\theta)\right) = \gamma^{(+)*}(\theta) \tag{36}$$

where $\gamma^{(+)^*}$ represents the complex conjugate of $\gamma^{(+)}$. The full ferromagnetic locus is then [13, 14]

$$\gamma(\theta) = \frac{1}{q-1}\left(-1+\sqrt{q}\mathrm{e}^{\mathrm{i}\theta}\right). \tag{37}$$

This circular locus is analogous to the circle theorem of Lee and Yang [9]. In the field-driven case, where one is interested in Lee–Yang zeros in the complex $z = \exp h$ plane (h is an external magnetic field), formulae analogous to (12)–(17) apply where e, c, etc are replaced by the corresponding derivative of the free energy with respect to h (the magnetization m, the susceptibility χ, etc). There, the partition function is unchanged under $h \to -h$ and consequently $\Delta\gamma_l = 0$ for even l. Therefore $\mathcal{L}(\theta) = 0$ and the locus of zeros is $z = \exp \mathrm{i}\theta$. This is Lee's proof of the Lee–Yang theorem [8]. One observes that considering $h \to -h$ as a self-duality map and identifying it with complex conjugation yields this locus.

4. The singular parts of the thermodynamic functions in the thermodynamic limit

From (11), the density of zeros in the temperature driven case is†

$$g(\theta) = \lim_{V\to\infty}\frac{1}{V}\frac{\mathrm{d}j}{\mathrm{d}\theta} = \frac{\Delta e}{2\pi}\left(1+\frac{1}{(q-1)\gamma(\theta)}\right)\left\{1+\frac{\Delta c}{k_B\beta_c^2\Delta e}\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right.$$
$$\left.+\frac{1}{2!}\frac{\Delta\gamma_3}{\Delta e}\left(\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right)^2+\frac{1}{3!}\frac{\Delta\gamma_4}{\Delta e}\left(\ln\left((\sqrt{q}+1)\gamma(\theta)\right)\right)^3+\cdots\right\}. \tag{38}$$

The internal energy is (from (3) or [22, 23])

$$e = \text{constanst} + u\int_0^{2\pi}\frac{g(\theta)}{u-\gamma(\theta)}\mathrm{d}\theta. \tag{39}$$

Therefore, from (37)–(39), the internal energy is

$$e(\beta<\beta_c) = e_0 \tag{40}$$

$$e(\beta>\beta_c) = e_0 - \Delta e\left\{1+\frac{\Delta c}{k_B\beta_c^2\Delta e}(\beta_c-\beta)+\frac{1}{2!}\frac{\Delta\gamma_3}{\Delta e}(\beta_c-\beta)^2\right.$$
$$\left.+\frac{1}{3!}\frac{\Delta\gamma_4}{\Delta e}(\beta_c-\beta)^3+\cdots\right\}. \tag{41}$$

When separate Fisher loci which do not cross the positive real temperature axis are accounted for, e_0 becomes a temperature-dependent quantity corresponding to the regular part of the internal energy near the transition point. At β_c the internal energy discontinuity $e(\beta = \beta_c^-) - e(\beta = \beta_c^+) = \Delta e$ is recovered. Appropriate differentiation recovers the discontinuities in specific heat and higher cumulants. Thus the locus (37) and the density (38) are sufficient to give the *singular* parts of the thermodynamic functions in the infinite-volume limit.

5. Conclusions

In summary, we have applied the duality transformation (1), under which the $d = 2$ q-state Potts model is invariant, to the Fisher zeros recently found by Lee [8] for systems with a first-order phase transition. The requirement that the dual of the locus of zeros be identical

† The existence of a unique density of zeros in the thermodynamic limit was proven by Salmhofer [24].

to the locus of the duals of the zeros order by order in the expansion parameter θ in the thermodynamic limit (i) recovers the ratio of specific heat to internal energy discontinuity at criticality and the relations between the discontinuities of higher cumulants and (ii) identifies duality with complex conjugation order by order in θ.

While we have verified the identification (ii) up to $\mathcal{O}(\theta^{10})$, the conjecture that it holds (for zeros governing ferromagnetic critical behaviour) to all orders gives that the full ferromagnetic locus is the circle (37) in the complex u-plane. The equation (37) was first conjectured by Martin [13] and Maillard and Rammal [14] on the basis of analogous Ising results [10]. The same conjecture, based on numerical results for small lattices was made in [15] and proven for *infinite* q in [16] (see also [17]). Thus while this idea has been around for a long time and supported by numerical results on small lattices, this identification of the dual of the locus with the locus of the duals places it on an analytic footing.

The locus (37), together with the density of zeros is sufficient to recover the *singular* parts of all thermodynamic functions in the thermodynamic limit. It is to be expected that the *regular* parts come from separate loci of zeros which do not cross the positive real temperature axis.

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